

A NOTE ON THE PROOF OF THEOREM 13 IN THE PAPER "GENERALIZED GYROVECTOR SPACES AND A MAZUR-ULAM THEOREM"

TOSHIKAZU ABE AND OSAMU HATORI

ABSTRACT. We give a revision of the proof of a Mazur-Ulam theorem for generalized gyrovector spaces given in [1].

1. A PROOF OF A MAZUR-ULAM THEOREM FOR GGV'S

We introduce a notion of the generalized gyrovector space and give a Mazur-Ulam theorem for the generalized gyrovector space as Theorem 13 and Corollary 14 in [1]. The essential part of this Mazur-Ulam theorem is Theorem 13, and it is exhibited without a required corection of the proof in the print version of [1] although the website version is revised. One may revise the proof easily, but for the sake of the convenience of the readers and the completeness of the proof we give a revision of the proof of Theorem 13. Notations and terminologies are due to [1]. The following is Theorem 13 in [1].

Theorem 1. *Let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be GGV's with ϱ_1 and ϱ_2 being gyrometrics of G_1 and G_2 , respectively. Suppose that $T : G_1 \rightarrow G_2$ is a gyrometric preserving surjection. Then T preserves the gyromidpoints;*

$$p(T\mathbf{a}, T\mathbf{b}) = Tp(\mathbf{a}, \mathbf{b})$$

for any pair $\mathbf{a}, \mathbf{b} \in G_1$.

Proof. Let $\mathbf{a}, \mathbf{b} \in G_1$ and \mathbf{z} be the gyromidpoint of \mathbf{a} and \mathbf{b} . Let W be the family of all bijective gyrometric preserving maps $S : G_1 \rightarrow G_1$ keeping the points \mathbf{a} and \mathbf{b} fixed, and set

$$(1) \quad \lambda = \sup\{f(\varrho(S\mathbf{z}, \mathbf{z})) : S \in W\} \in [0, \infty],$$

where f is the bijection which satisfies (F1) and (F2) of Proposition 18 in [1]. Note that in [1] the function f in Proposition 18 is denoted

1991 *Mathematics Subject Classification.* 47B49, 46L05, 51M10.

Key words and phrases. gyrovector spaces, a Mazur-Ulam theorem.

by $f : \|G\| \rightarrow \mathbb{R}$, but it reads as $f : \|\phi(G)\| \rightarrow \mathbb{R}$. For $S \in W$ we have $\varrho(Sz, \mathbf{a}) = \varrho(Sz, S\mathbf{a}) = \varrho(\mathbf{z}, \mathbf{a})$, hence

$$(2) \quad \varrho(Sz, \mathbf{z}) \leq \varrho(Sz, \mathbf{a}) \oplus'_1 \varrho(\mathbf{a}, \mathbf{z}) = 2 \otimes'_1 \varrho(\mathbf{a}, \mathbf{z}),$$

hence $f(\varrho(Sz, \mathbf{z})) \leq f(2 \otimes'_1 \varrho(\mathbf{a}, \mathbf{z})) = 2f(\varrho(\mathbf{a}, \mathbf{z}))$ which yields $\lambda < \infty$.

Let $\psi(\mathbf{x}) = 2 \otimes_1 \mathbf{z} \ominus_1 \mathbf{x}$ on G_1 . If $S \in W$, then so also is $S^* = \psi S^{-1} \psi S$, and therefore $f(\varrho(S^* \mathbf{z}, \mathbf{z})) \leq \lambda$. Since S^{-1} is a gyrometric preserving map, (p5) of Proposition 16 and Proposition 18 in [1] together imply that

$$(3) \quad \begin{aligned} \lambda &\geq f(\varrho(S^* \mathbf{z}, \mathbf{z})) = f(\varrho(\psi S^{-1} \psi S \mathbf{z}, \mathbf{z})) \\ &= f(\varrho(S^{-1} \psi S \mathbf{z}, \mathbf{z})) \\ &= f(\varrho(\psi S \mathbf{z}, S \mathbf{z})) \\ &= f(2 \otimes'_1 \varrho(S \mathbf{z}, \mathbf{z})) \\ &= 2f(\varrho(S \mathbf{z}, \mathbf{z})) \end{aligned}$$

for all $S \in W$, showing that $\lambda \geq 2\lambda$. Thus $\lambda = 0$, which means that $Sz = \mathbf{z}$ for all $S \in W$ since f is linear and bijective.

Let $T : G_1 \rightarrow G_2$ be a bijective gyrometric preserving map. Let \mathbf{z}' be the gyromidpoint of $T(\mathbf{a})$ and $T(\mathbf{b})$. To prove the theorem we must show that $T(\mathbf{z}) = \mathbf{z}'$. Let $\psi'(\mathbf{y}) = 2 \otimes_2 \mathbf{z}' \ominus_2 \mathbf{y}$ on G_2 . Then the map $\psi T^{-1} \psi' T$ is in W , whence $\psi T^{-1} \psi' T(\mathbf{z}) = \mathbf{z}$. This implies that $\psi'(T(\mathbf{z})) = T(\mathbf{z})$. Due to (p3) of Proposition 16, we obtain $T(\mathbf{z}) = \mathbf{z}'$. \square

REFERENCES

- [1] T. Abe and O. Hatori, *Generalized gyrovector spaces and a Mazur-Ulam theorem*, Publ. Math. Debrecen, **87** (2015), 393–413

DEPARTMENT OF INFORMATION ENGINEERING, FACULTY OF ENGINEERING,
NIIGATA UNIVERSITY, JAPAN

E-mail address: abebin08@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY,
JAPAN

E-mail address: hatori@math.sc.niigata-u.ac.jp